# International Journal For Academic Research & Development

# Vol.1 (2020)

# (Multidisciplinary)

# Curvature Tensors of a Finsler Space

Singh Brijendra Krishna
Department of Mathematics, Amity University Chhattisgarh, Raipur
bks0509@gmail.com

**Abstract**

*In the present theoretical analysis, It is defined as the vector space, whose metric is defined by a function, which satisfies the conditions of being a Finsler space. Just as a Riemannian space is locally Euclidean a Finsler space may be regarded as locally Minkowskian space.*

**Keywords:-***Tensor, Finisler Space, Minkowskian space. connections and covariant*

## 1. Introduction

A system of n quantities $X^i$ whose transformation law under (1.1) is equivalent to that of the $\dot{x}^{i'}$ is called a contra variant vector attached to the point $P(x^i)$ of $X_n$. Such contra variant vector constitutes the elements of our new vector space. Hence the totality of all contra variant vectors attached to $P(x^i)$ of $X_n$ is the tangent space denoted by $T_n(P)$ or $T_n(x^i)$ .

Further, since the transformation (3.2b) is homogeneous, we may regard the tangent space as 'Centred' affine spaces, the centre or origin corresponding to the values $\dot{x}^1 = 0, \dot{x}^2 = 0,...., \dot{x}^n = 0.$

**Indicatrix**

We consider that the function $F(x^i, \dot{x}^i)$ is defined for all the line elements $(x^i, \dot{x}^i)$ over the region $R$ of $X_n$. The equation $F(x^i, \dot{x}^i) = 1$ ( $x^i$ fixed, $\dot{x}^i$ variable) represent a $(n-1)$ dimensional locus in $P$ i.e. a hyper surface. This hyper surface plays the role of the unit sphere in the geometry of the vector space $T_n(P)$ and is called indicatrix.

**Minkowskian Space**

It is defined as the vector space, whose metric is defined by a function, which satisfies the conditions *A* to *C* of being a Finsler space. Just as a Riemannian space is locally Euclidean a Finsler space may be regarded as locally Minkowskian space.

**Metric Tensor**

With the help of equations (3.2a), we can easily see that the set of quantities $g_{ij}(x^i, \dot{x}^i)$ defined by the equation (2.6) form the components of a covariant tensor of rank 2. Also it is clear that $g_{ij}(x^i, \dot{x}^i)$ are positively homogeneous of degree zero in $\dot{x}^i$ and are symmetric in its indices *i* and *j*.

Because of the homogeneity condition (A) for the function $F(x^i, \dot{x}^i),$ we have

$$F^2(x, \dot{x}) = g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j. \qquad (1.1)$$

The inverse of $g_{ij}$ denoted by $g^{ij}$ is defined as

$$g_{ij}(x,\dot{x})\, g^{ik}(x\dot{x}) = \delta^k_j = \begin{cases} 1 & \text{if } k=i \\ 0 & \text{if } k\neq j \end{cases} \tag{1.2}$$

where $\delta^k_j$ is the well known kroncker delta.

### Definition

The tensor with covariant components $g_{ij}$ and contra variant components $g^{ij}$ is called the metric tensor or the first fundamental tensor of the Finsler space $F_n$.

The tensor $C_{ijk}(x,\dot{x})$ defined by

$$C_{ijk}(x,\dot{x}) \overset{def}{=} \frac{1}{2}\frac{\partial g_{ij}(x,\dot{x})}{\partial \dot{x}^k} \overset{def}{=} \frac{1}{4}\frac{\partial^3 F^3(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k} \tag{1.3}$$

is positively homogeneous of degree-1 in $\dot{x}^i$ and is symmetric in all three of its indices. This tensor is known as Cartan's *C*-tensor and satisfy the following conditions

$$C_{ijk}(x,\dot{x})\dot{x}^i = C_{ijk}(x,\dot{x})\dot{x}^j = C_{ijk}(x,\dot{x})\dot{x}^k = 0 \tag{1.4}$$

and

$$\frac{\partial C_{ijk}(x,\dot{x})}{\partial x^h}\dot{x}^i = \frac{\partial C_{ijk}(x,\dot{x})}{\partial x^h}\dot{x}^j = \frac{\partial C_{ijk}(x,\dot{x})}{\partial x^h}\dot{x}^k = 0. \tag{1.5}$$

## 2. Dual Tangent Space

Corresponding to each contra variant vector $\dot{x}^i$ of $T_n(P)$, we have may associate a covariant vector $\dot{x}_i$ defined by the relation

$$\dot{x}_i = g_{ij}(x,\dot{x})\dot{x}^j \tag{2.1}$$

where it may be noted that the directional argument in the $g_{ij}$ coincides with the vector $\dot{x}^i$ under consideration.

The totality of all covariant vectors $\dot{x}_i$ given by the equation (2.1) associated to a point *P*, is called the dual tangent space of $X_n$ at *P* and is denoted by $T'_n(P)$.

### Magnitude of A Vector, Angle and Orthogonality

Let $X^i$ be a vector and $\dot{x}^i$ be an arbitrary fixed direction. Then the scalar $|X|$ is given by

$$|X|^2 = g_{ij}(x,\dot{x})\, X^i X^j \tag{3.1}$$

is called the square of the magnitude of the vector $X^i$ for the pre assigned direction $\dot{x}^i$. The Minkowskian magnitude of a vector $X^i$ is defined as

$$|X|^2 = g_{ij}(x^i, X^i)\, X^i X^j. \tag{3.2}$$

The Minkowskian cosine corresponding to two arbitrary directions $\lambda^i$ and $\mu^i$ is defined by the ratio

$$\cos(\lambda,\mu) = \frac{g_{ij}(x^k,\lambda^k)\lambda^i\mu^j}{F(x^k,\lambda^k)F(x^k,\mu^k)}. \tag{3.3}$$

From the ratio (3.3), the Minkowskian cosine is not symmetrical in its arguments $\lambda^i$ and $\mu^i$.

However, for an arbitrary fixed direction $\dot{x}^i$, the cosine for two directions $\lambda^i, \mu^i$ can be written as

$$\cos(\lambda,\mu) = \frac{g_{ij}(x^k,\dot{x}^k)\lambda^i\mu^j}{\left[g_{ij}(x^k,\dot{x}^k)\lambda^i\lambda^j\right]^{1/2}\left[g_{ij}(x^k,y^k)\mu^i\mu^j\right]^{1/2}}. \tag{3.4}$$

This expression is symmetric in $\lambda$ and $\mu$ but it depends on the original choice of the direction $\dot{x}^i$. Therefore, it is basically different from (3.3).

**Definition (3.1):**

The vector $\mu^i$ is said to be orthogonal with respect to the vector $\lambda^i$ iff

$$g_{ij}(x^k,\lambda^k)\lambda^i\mu^j = 0. \tag{3.5}$$

Thus, we see that orthogonality is not a symmetrical relationship between the two vectors $\lambda^i$ and $\mu^i$.

**Definition (3.2):**

The vector $\lambda^i$ and $\mu^i$ are called orthogonal (for a pre assigned direction $\dot{x}^i$) iff

$$g_{ij}(x^k,\dot{x}^k)\lambda^i\mu^j = 0. \tag{3.6}$$

This definition of orthogonality is symmetrical in $\lambda^i$ and $\mu^i$.

**Connection and Covariant Differentiation in $F_n$:**

The connection theories of Finsler space $F_n$ have been studied by many authors. These theories may broadly be divided into two types. In one $F_n$ is constructed of the line elements and is used by most of the researchers [17] [5] [2] [13] and the other is derived term Minkowkian tangent spaces [11] [3].

**(A) Finsler Connection**

The Finsler connection $F\Gamma$ of a Finsler space $F_n$ is a triad $(F^i_{jk}, N^i_k, C^i_{jk})$ of a V-connection $F^i_{jk}$, a non linear connection $N^i_k$ and a vertical connection $C^i_{jk}$ [17]. In general, the vertical connection $C^i_{jk}$ is different from Cartan's $C$-tensor obtained from $C_{ijk}$ given by the equation (1.3). However, there are certain Finsler connections to be discussed, in which two quantities (vertical connections and Cartan's $C$-tensor) are identical.

If a Finsler connection is given, the $h$- and $v$-covariant derivatives of any tensor field $T^i_j$ are defined as

$$T^i_{j|k} = \partial_k T^i_j + T^m_j F^i_{mk} - T^i_m F^m_{jk} \tag{4.1}$$

and

$$T^i_{j|k} = \dot{\partial}_k T^i_j + T^m_j C^i_{mk} - T^i_m C^m_{jk} \tag{4.2}$$

respectively, where

$$d_k = \partial_k - N^m_k \dot{\partial}_m, \tag{4.3}$$

$$\partial_k = \frac{\partial}{\partial x^k}, \quad \dot{\partial}_k = \frac{\partial}{\partial \dot{x}^k},$$

$\left(|_k\right)$ and $\left(\|_k\right)$ denotes the $h$ and $v$-covariant derivatives respectively.

For any Finsler connection $(F_{jk}^i, N_k^i, C_{jk}^i)$ we have five tensors which are expressed as follows:

The $(h)h$-torsion tensor:

$$T_{jk}^i = F_{jk}^i - F_{kj}^i, \tag{4.4}$$

The $(v)V$-torsion tensor:

$$S_{jk}^i = C_{jk}^i - C_{kj}^i, \tag{4.5}$$

The $(h)hv$-torsion tensor: $C_{jk}^i = $ as the connection $C_{jk}^i$, $\tag{4.6}$

The $(v)h$-torsion tensor: $R_{jk}^i = d_k N_j^i - d_j N_k^i$, $\tag{4.7}$

The $(v)hv$-torsion tensor: $P_{jk}^i = \dot{\partial}_k N_j^i - F_{kj}^i$. $\tag{4.8}$

The deflection tensor field $D_j^i$ of a Finsler connection is given by

$$D_j^i = \dot{x}^k N_j^i - F_{kj}^i. \tag{4.9}$$

When a Finsler metric is given, various Finsler connections may be defined from the metric. The well known examples are the Rund connection, the Cartan connection and the Berwald connection which are given below.

**(B) Rund Connection**

As in Riemannian geometry, the Christoffel's symbols of first and second kinds have been defined as

$$\gamma_{hij}(x, \dot{x}) = \frac{1}{2}(\partial_j g_{hi} + \partial_h g_{ij} - \partial_i g_{jh}) \tag{4.10}$$

and

$$\gamma_{ij}^h(x, \dot{x}) = g^{hk}(x, \dot{x})\gamma_{ikj}(x, \dot{x}). \tag{4.11}$$

From the definition it is clear that $\gamma_{ikj}(x, \dot{x})$ is symmetric in its extreme indices and $\gamma_{ij}^h(x, \dot{x})$ is symmetric in its lower indices and satisfy the relation

$$\partial_k g_{ij}(x, \dot{x}) = \gamma_{ijk}(x, \dot{x}) + \gamma_{jik}(x, \dot{x}). \tag{4.12}$$

The symbols $\Gamma_{ij}^h(x, \dot{x})$ are defined as

$$\Gamma_{ij}^h(x, \dot{x}) = \gamma_{ij}^h(x, \dot{x}) - C_{im}^h(x, \dot{x})\gamma_{kj}^m(x, \dot{x})\dot{x}^k \tag{4.13}$$

where

$$C_{ij}^h(x, \dot{x}) = g^{hk}(x, \dot{x})C_{ikj}(x, \dot{x}) \tag{4.14}$$

and Cartan's $C$-tensor $C_{ikj}$ is defined by (1.3).

For a vector $X^i$ the components $\dfrac{\delta X^i}{\delta t}$ defined by

$$\frac{\delta X^i}{\delta t} = \frac{d X^i}{d t} + \Gamma_{jk}^i(x, \dot{x}) X^j \frac{dx^k}{dt} \tag{4.15}$$

form the contra variant components of a vector. The process of differentiation given by (4.15) is called '$\delta$-differentiation'.

In particular, this process gives a well defined parallel displacement. The vector $X^i + dX^i$ of $T_n(x^i + dx^i)$ is said to be obtained from the vector $X^i$ of $T_n(x^i)$ by parallel displacement if $\delta X^i = 0$. Hence, for such a displacement, we have [21]

$$dX^i = -\Gamma_{jk}^i(x, \dot{x}) X^j dx^k \tag{4.16}$$

The partial $\delta$ - derivative with respect to $x^k$ in the direction $\dot{x}^i$ of the arbitrary tensor $T_j^i(x, \xi)$ is de-

fined by the formula [21]

$$T^i_{j;k} = \partial_k T^i_j + \dot{\partial}_h T^i_j \partial_k \xi^h + T^m_j \Gamma^{*i}_{mk}(x,\dot{x}) - T^i_m \Gamma^{*m}_{jk}(x,\dot{x}),$$ (4.17)

where the coefficients $\Gamma^{*m}_{jk}(x,\dot{x})$ is given by

$$\Gamma^{*m}_{jk}(x,\dot{x}) = g^{ih}(x,\dot{x})\Gamma^{*m}_{jk}(x,\dot{x})$$ (4.18)

and

$$\Gamma^*_{jhk}(x,\dot{x}) = \gamma_{jhk}(x,\dot{x}) - [C_{khi}(x,\dot{x})\Gamma^i_{jm}(x,\dot{x}) +$$
$$+ C_{hji}(x,\dot{x})\Gamma^i_{km}(x,\dot{x}) - C_{jki}(x,\dot{x})\Gamma^i_{hm}(x,\dot{x})]\dot{x}^m.$$ (4.19)

The symbol $\Gamma^{*i}_{jk}$ is symmetric in its lower indices $j$ and $k$, while $\Gamma^i_{jk}$ is no-symmetric in $j$ and $k$. Also, we have

$$\Gamma^{*i}_{jk}\dot{x}^j\dot{x}^k = \Gamma^i_{jk}\dot{x}^j\dot{x}^k = \gamma^i_{jk}\dot{x}^j\dot{x}^k,$$ (4.20)

$$\Gamma^i_{jk}\dot{x}^k = \Gamma^{*i}_{jk}\dot{x}^k,$$ (4.21)

$$\Gamma^i_{jk}\dot{x}^j = \gamma^i_{jk}\dot{x}^j.$$ (4.22)

The partial $\delta$-derivative of the metric tensor $g_{ij}(x,\xi)$ in the direction $\dot{x}^i$ in view of (4.17) is given by

$$g_{ij}(x,\xi);_k = \partial_k g_{ij}(x,\xi) + 2C_{ijh}(x,\xi)\partial_k \xi^h$$
$$- g_{hj}(x,\xi)\Gamma^{*h}_{ik}(x,\dot{x}) - g_{ih}(x,\xi)\Gamma^{*h}_{jk}(x,\dot{x})$$ (4.23)

If, in particular, $\dot{x}^i = \xi^i$, the above equation reduces to

$$g_{ij}(x,\xi);_k = 2C_{ijk}(x,\xi)\xi^h_{;k}.$$ (4.24)

We see that the partial $\delta$-derivative of the metric tensor $g_{ij}$ does not vanish in general. Therefore, further developments of theory of Finsler spaces will differ considerably from the established results of Riemannian geometry in which the covariant derivative of the metric tensor vanishes.

Further, it is to be noted that if the vector field $\xi^i$ is stationary, and that is $\xi^i_{;j} = 0$ then the partial $\delta$-differentiation of a tensor field is $h$-covariant derivative with respect to the Rund connection $(\Gamma^{*i}_{jk}, G^i_j, 0)$ where $\Gamma^{*i}_{jk}$ is V-connection defined by the equation (4.19) and $G^i_j$ is defined by

$$G^i_j(x,\dot{x}) = \dot{\partial}_j G^i, \; 2G^i(x,\dot{x}) = \gamma^i_{jk}\dot{x}^j\dot{x}^k$$ (4.25)

and the vertical connection $C^i_{jk}$ vanishes in this triad. Hence the $v$-covariant derivative of a tensor field is identical to the partial derivative with respect to the element of support $\dot{x}^i$ [17] [21].

## Conclusion

We see that the partial $\delta$-derivative of the metric tensor $g_{ij}$ does not vanish in general. Therefore, further developments of theory of Finsler spaces will differ considerably from the established results of Riemannian geometry in which the covariant derivative of the metric tensor vanishes.
When a Finsler metric is given, various Finsler connections may be defined from the metric. The well known examples are the Rund connection, the Cartan connection and the Berwald connection which are given below.

## References
1.Adati, T and Yamaguchi, S. *On some transformations in Riemannian recurrent space, THU Math. 3, 8-12, (1967).*

2. Adat, T. and Miyazawa, T., *On a Riemannian space with recurrent conformal curvature, Tensor(N. S.), 18, 348-354, (1966).*

3. Amur, K. S., *Generalzed geodesic torsin, Tensor(N. S.) 12(2), 133-139, (1962).*

4. Amur, K. S., *A note on deformation of a surface along the rays of a rectilinear congruence, Tensor(N. S.) 16(1), 56-59, (1965).*

5. Amur, K. S., *On the curvatures of a vector field on a surface in relation to a congruence of curves, Tensor(N. S.) 14(1), 6-17, (1963)*

6. Ancochea, G., *Kovariante Differentiation und die Ricerschen Identitation in Finsler Raumen, Rev. Mat. hisp. Amer. 2,5,8, 261-264, (1933).*

7. Aronezaju, N., *Sur quelques problems concerrnan les espaces de Minkowski spaces vectorieles generaux, Atti. Accad. Naz. Lincei Rend. (6)26, 374-376, (1937).*

8. Auslander, L., *On curvature in Finsler geoemtry, Trans. Amer. Math. Soc. 79, 378-388, (1955).*

9. Berwald, L., *Invarianten bei der variation vielfacher integrale und parallel hyperflachen in Cartanschem Raumen, Composite Math. 7, 141-176, (1939).*

10. Berwald, L., *Untersuchung der krummung allgemeiner metrischer Raume sufground des in inhen herra schenden parallelismus, Math.Z. 25, 40-76, (1926), Correction Math.Z. 26, pp. 176, (1927).*

11. Berwald, L., *Uker die n-dimensionalen Geometrien kronstanter Krummung in denen de geradedie Kurzestan sind, Math. Z. 30, 449-469, (1929).*

12. Barka, Bandana, *On an associated Riemannian space, Prog. Math. 1, 1-4, (1967).*

13. Beem John, K., *Indefinite Finsler spaces and time like spaces, Conad. J. Math. 22, 1035-1039, (1970).*

14. Bliss, G. A., *A generalization of the notion of angle, Trans. Amer. Math. Soc. 27, 61-67, (1925).*

15. Eliznikas, V. I., *Finsler spaces their generalizations, Algerbra Topology, Geoemtry, 73-125, (1967).*

16. Bianchi, L., *Lezioni di Gemetria differenziale, Boflogna (1924).*

17. Bolza, O., *Vorlesungen uber variation srechung Lei pzig (1909),*